PHYS4150 — PLASMA PHYSICS

LECTURE 9 - SLOW E FIELD VARIATIONS, POLARIZATION DRIFT, AND PLASMAS AS A DIELECTRIC

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1 POLARIZATION DRIFT

Let us return to the case of uniform orthogonal electric and magnetic fields which we have studied in lecture 6. This time we allow for slow variations of the electric field. Like in lecture 2 we start with the particle's equation of motion, but this time we take the crossproduct with $\frac{\mathbf{B}}{B^2}$

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad \Big| \times \frac{\mathbf{B}}{B^2},$$

and

Remember that $(A\times B)\times C = (A\cdot C)B - (B\cdot C)A$

$$\frac{m}{q}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\times\frac{\mathbf{B}}{B^2} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + (\mathbf{v}\times\mathbf{B})\times\frac{\mathbf{B}}{B^2}$$

$$\frac{m}{q}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\times\frac{\mathbf{B}}{B^2} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}}{B^2}(\mathbf{v}\cdot\mathbf{B}) - \mathbf{v}$$

or after rearranging and using that $\dot{\mathbf{B}} = 0$

$$\underbrace{\mathbf{v} - \frac{\mathbf{B}}{B^2}(\mathbf{v}\cdot\mathbf{B})}_{\perp \text{ velocity}} = \underbrace{\frac{\mathbf{E}\times\mathbf{B}}{B^2}}_{\mathbf{E}\times\mathbf{B} \text{ drift}} - \frac{m}{q}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\times\frac{\mathbf{B}}{B^2} = \mathbf{v}_E - \frac{m}{qB^2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\times\mathbf{B})\,.$$

The left side can be interpreted as a perpendicular velocity and the first term on the right side is again the $\mathbf{E}\times\mathbf{B}$ drift. Now we average over one gyroperiod

$$\mathbf{v}_d = \mathbf{v}_E - \frac{m}{qB^2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\times\mathbf{B})$$

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and identify the left side as a drift velocity. In lecture 6 we have found that the $\mathbf{E} \times \mathbf{B}$ drift results from a Lorentz transformation $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ into the particle's moving reference frame and the fact that $\mathbf{E}'$ must vanish for a free particle. Similarily, the $\mathbf{v} \times \mathbf{B}$ term represents a perpendicular electric field $\mathbf{E}_\perp = -\mathbf{v} \times \mathbf{B}$, and thus

$$\mathbf{v}_d = \mathbf{v}_E + \frac{m}{qB^2}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t} = \mathbf{v}_E + \frac{q}{|q|}\frac{1}{\omega_c B}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}.$$

The last term is called *Polarization drift*

$$\boxed{\mathbf{v}_p = \frac{q}{|q|}\frac{1}{\omega_c B}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t},}$$

which results from a slow variation of the electric field. $\mathbf{v}_p$ is proportional to the particle mass and depends on its polarity. Thus, the polarization drift creates a current into the direction of the electric field

$$\mathbf{j}_p = n_0 e(\mathbf{v}_{pi} - \mathbf{v}_{pe}) = \frac{n_0(m_e + m_i)}{B^2}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}.$$

2 DIELECTRIC CONSTANT OF A PLASMA

A dielectric material is an insulator which gets polarized by an electric field. Clearly we expect such a behavior from a plasma. To see this, lets position a plasma between the plates of a capacitor

$$C = (\epsilon_0 + \epsilon_p)\frac{A}{d},$$

where A is the area of the plates separated by d. $\epsilon_0 = 8.854\,\mathrm{F\,m^{-1}}$ is the permittivity of the vacuum, and ϵ_p is the permittivity of the plasma we are interested in. When applying an alternating voltage V at the capacitor, we will observe the current

$$i = C\frac{\partial V}{\partial t},$$

which after substituting C into it and expressing V/d by E

$$i_p = \epsilon_p A\frac{1}{d}\frac{\partial V}{\partial t} = \epsilon_p A\frac{\partial E}{\partial t}.$$

On the other hand, in the previous section we have just found the current resulting from an alternating electric field applied to a plasma

$$i_p = j_p A = \frac{\rho_m}{B^2}A\frac{\partial E}{\partial t}.$$

Comparing the two expressions for i_p yields the *low frequency plasma permittivity for transverse motion*

$$\epsilon_p = \frac{\rho_m}{B^2},$$

where ρ_m is the plasma's mass density. We get further insight into the nature of ϵ_p by substituting characteristic plasma parameters into the relative permittivity

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = k = 1 + \frac{\rho_m}{\epsilon_0 B^2},$$

where ϵ is the total permittivity

$$\epsilon = \epsilon_0 + \frac{\rho_m}{B^2} = \epsilon_0 \left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right).$$

Now,

$$\begin{aligned}
\epsilon_r &\approx 1 + \frac{n m_i}{\epsilon_0 B^2} = 1 + \frac{e^2}{e^2}\frac{m_e}{m_e}\frac{n m_i}{\epsilon_0 B^2} \\
&\approx 1 + \underbrace{\left(\frac{n e^2}{\epsilon_0 m_e}\right)}_{\omega_p^2} \underbrace{\left(\frac{m_i}{e B}\right)}_{1/\omega_{c,i}} \underbrace{\left(\frac{m_e}{e B}\right)}_{1/\omega_{c,e}}
\end{aligned}$$

and hence,

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\omega_p^2}{\omega_{c,i}\omega_{c,e}}.} \tag{1}$$

The low frequency plasma permittivity depends only on the plasma frequency and the cyclotron frequencies of the ions and electrons. We will derive later the same expression more rigorously using the fluid description of the plasma. This will also elucidate the meaning of the simplifications we have made to obtain Eq. (1).

3 WHERE DO WE STAND?

Parameters	Drifts	Drift currents
$\rho_c = \frac{m v_\perp}{\lvert q\rvert B}$	$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$	
$\omega_c = \frac{\lvert q\rvert B}{m}$	$\mathbf{v}_p = \frac{1}{\omega_p B}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}$	$j_p = \frac{n_e(m_i + m_e)}{B^2}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}$
$\mu = \frac{T_\perp}{B}$	$\mathbf{v}_G = \frac{T_\perp}{qB}\left[\frac{\hat{\mathbf{B}}\times\nabla\mathbf{B}}{B}\right]$	$j_G = \frac{n_e(m_i + m_e)}{B^2}(\mathbf{B}\times\nabla\mathbf{B})$
	$\mathbf{v}_c = \frac{2T_\parallel}{qB}\left[\frac{\hat{\mathbf{B}}\times\hat{\mathbf{R}}_c}{R_c}\right]$	$j_R = \frac{n_e(T_\parallel^i + T_\parallel^e)}{B^2 R_c^2}(\mathbf{B}\times\mathbf{R}_c)$

INVARIANTS

1. μ

2. $J = m\int \mathbf{v}_{\|}\,\mathrm{d}s$

3. $\Phi = \oint v_d r\,\mathrm{d}\phi = \frac{2\pi m}{q^2}M$

4 TRAPPING IN DIPOLAR MAGNETIC FIELDS

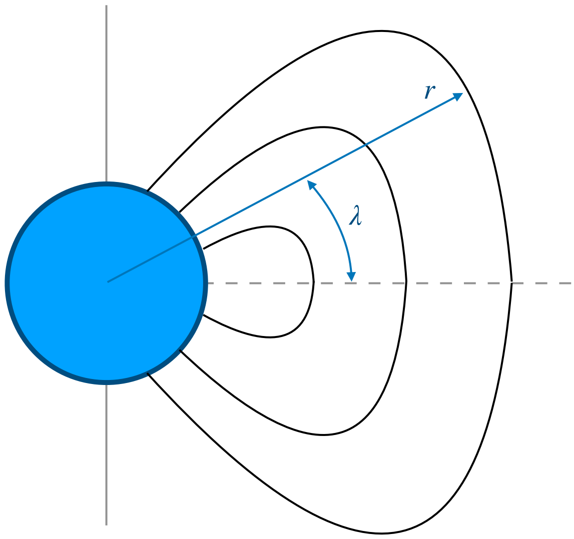
4.1 *Dipole field*

$$\mathbf{B}_{dipole} = \frac{\mu_0}{4\pi r^2}\frac{M}{r}\left(-2\sin\lambda\hat{\mathbf{e}}_r + \cos\lambda\hat{\mathbf{e}}_\lambda\right) \tag{2}$$

and

$$B_{dipole} = \frac{\mu_0}{4\pi r^2}\frac{M}{r}\left(1+3\sin^2\lambda\right)^{1/2}, \tag{3}$$

where λ is the *magnetic latitude* and $M = 8.05\cdot 10^{22}\,\mathrm{A\,m^2}$ is the magnetic moment of the Earth' dipolar field.



We need an equation for a magnetic field line, i.e. an expression for $B(\lambda)$ along a *line of force*. If $\mathrm{d}\mathbf{s} = (\mathrm{d}r, \lambda\,\mathrm{d}\lambda)$ is an arc element, then a line of force satisfies

$$\mathrm{d}\mathbf{s}\times\mathbf{B} \stackrel{!}{=} 0,$$

from which follows that

$$\frac{\mathrm{d}r}{B_r} = \frac{r\,\mathrm{d}\lambda}{B_\lambda}.$$

Using Eq. (2)

$$\frac{\mathrm{d}r}{r} = -\frac{2\sin\lambda\,\mathrm{d}\lambda}{\cos\lambda} = \frac{2\mathrm{d}(\cos\lambda)}{\cos\lambda},$$

and after integration

$$r(\lambda) = r(0)\cos^2\lambda. \tag{4}$$

$r(0)$ is usually expressed as multiple $L = r(0)/R_p$ of the planet's equatorial radius R_p. Substituting Eq. (4) into Eq. (3) and introducing the equatorial field strength at the planet's surface

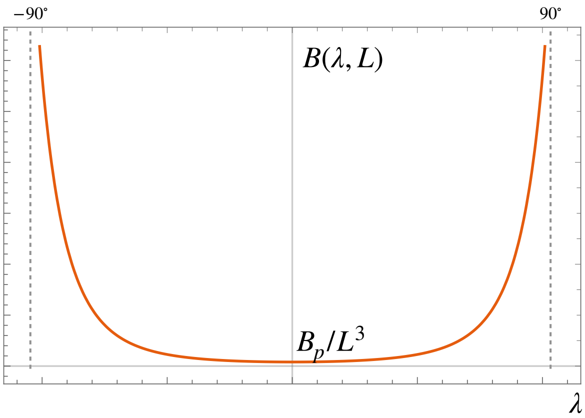
$$B_p = \frac{\mu_0 M_p}{4\pi R_p^3}$$

gives the standard form for a planetary magnetic dipolar field

$$B(\lambda, L) = \frac{B_p}{L^3}\frac{\left(1+3\sin^2\lambda\right)^{1/2}}{\cos^6\lambda} \tag{5}$$

$$\cos^2\lambda_p = L^{-1}, \tag{6}$$

where λ_p is the latitude at which a field line of a given L dissects the planet's surface.



As apparent from the figure above, a planetary dipolar field is in fact a magnetic bottle configuration.